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Static and Time-Dependent Perturbations of the Classical Elliptical Billiard

Jair Koiller,[1] Roberto Markarian,[2] Sylvie Oliffson Kamphorst,[3] and Sônia Pinto de Carvalho[3]

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The elliptical billiard problem defines a two-dimensional integrable discrete dynamical system. Integrability not being a robust property, we study some static and time-dependent perturbations of this problem. For the static case, we observe the transition from integrability to chaos, on some perturbations of the ellipse. Then we study time-dependent perturbations, supposing that the boundary deforms periodically with the time, remaining always an ellipse. We investigate numerically the now four-dimensional phase space, looking mainly at the question of whether or not the velocity of a given trajectory may increase indefinitely.

KEY WORDS: Classical elliptical billiard; static and time-dependent perturbations.

1. THE CLASSICAL ELLIPTICAL BILLIARD

The classical elliptical billiard problem consists in the study of the free motion of a point particle inside the plane region bounded by an ellipse, being reflected elastically at the impacts with the boundary.

[1] Laboratório Nacional de Computação Científica and Instituto de Matemática, Rua Lauro Muller 455, 22290-160 Rio de Janeiro, Brazil; E-mail: jair@lncc.br.

[2] Instituto de Matemática y Estadística "Prof. Ing. Rafael Laguardia," Faculdad de Ingeniería, Montevideo, Uruguay; E-mail: roma@fing.edu.uy.

[3] Departamento de Matemática, ICEx, UFMG, 30161-970, Belo Horizonte, Brazil; E-mail: syok@mat.ufmg.br, sonia@mat.ufmg.br.

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One of the possible ways to construct a mathematical model for this problem is to define an ellipse, in Cartesian coordinates, by $f(x, y) = x^2/a^2 + y^2 = 1$, $a > 1$, and a Lagrangian system $L = T - V$ by

$$T(x', y') = \tfrac{1}{2}(x'^2 + y'^2)$$

$$V(x, y) = \lim_{N \to \infty} [f(x, y)]^N = \begin{cases} 0 & \text{inside the ellipse} \\ \infty & \text{outside the ellipse} \end{cases}$$

Suppose that the particle leaves the ellipse from $q_0 = (x(\varphi_0), y(\varphi_0))$ with velocity $\mathbf{v}_0$. It will then ove with constant velocity on a straight line until it reaches the ellipse again at $q_1 = (x(\varphi_1), y(\varphi_1))$, with incoming velocity $\mathbf{v}_0$ and outgoing velocity $\mathbf{v}_1$. Since the system is Lagrangian, the conservation of energy and momentum give that $\mathbf{v}_1 = \mathbf{v}_0 - 2\langle \mathbf{v}_0, \mathbf{n}_1 \rangle \mathbf{n}_1$, where $\mathbf{n}_1$ is the unitary external normal vector at q_1 and $\|\mathbf{v}_1\| = \|\mathbf{v}_0\|$. We can take $\|\mathbf{v}_i\| = 1$ and then $\mathbf{v}_i$ is defined by the angle α_i it makes with the unitary tangent vector.

Since the ellipse is a Jordan curve, $f(x, y) = 1$ constitutes a section, transverse to the flow defined by the Lagrangian L. Let us take the angle φ between the x axis and the oriented tangent to the ellipse as a parameter.

The billiard map T is defined on the phase space $\mathscr{A} = [0, 2\pi) \times (0, \pi)$, with coordinates φ and α. Given an initial condition (φ_0, α_0), $(\varphi_1, \alpha_1) = T(\varphi_0, \alpha_0)$ is given by

$$f(q_0 + \mathbf{v}_0 \bar{t}) = 1, \qquad \bar{t} > 0$$

$$q_1 = q_0 + \mathbf{v}_0 \bar{t}$$

$$\alpha_1 = \varphi_1 - (\varphi_0 + \alpha_0)$$

where $\mathbf{v}_0 = (\cos(\varphi_0 + \alpha_0), \sin(\varphi_0 + \alpha_0))$.

As the ellipse is C^∞, the map $T: \mathscr{A} \to \mathscr{A}$ is a C^∞-diffeomorphism, preserving the area $d\mu = R \sin \alpha \, d\alpha \, d\varphi$, where $R(\varphi) = a^2/(a^2 \sin^2 \varphi + \cos^2 \varphi)^{3/2}$ is the radius of curvature at $(x(\varphi), y(\varphi))$.

Its derivative is

$$DT_{(\varphi_0, \alpha_0)} = \frac{1}{R_1 \sin \alpha_1} \begin{pmatrix} l_{01} - R_0 \sin \alpha_0 & l_{01} \\ l_{01} - R_0 \sin \alpha_0 - R_1 \sin \alpha_1 & l_{01} - R_1 \sin \alpha_1 \end{pmatrix}$$

where l_{01} is the distance between q_0 and q_1, and $R_i = R(\varphi_i)$, $i = 0, 1$.

This billiard system is integrable. The function

$$F(\varphi, \alpha) = \frac{\cos^2 \alpha - e^2 \cos^2 \varphi}{1 - e^2 \cos^2 \varphi}$$

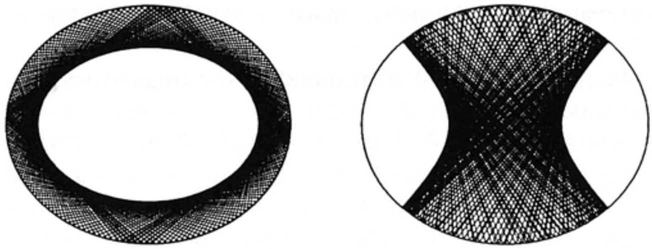

Fig. 1. Conical caustics.

where $e = (a^2 - 1)^{1/2}/a$ is the eccentricity of the ellipse, is a first integral for T, i.e., F is constant on the orbits under T. (Actually, Birkhoff conjectured that the only convex integrable billiard is the ellipse.)

Physically, F may be interpreted as the product of the angular momenta about the two foci.[2] This product is conserved along a trajectory since the movement is uniform between two impacts, there is a reflection at the ellipse, and the focal radii make equal angles with the tangent to the ellipse.

Geometrically, $F(\varphi_0, \alpha_0)$ has a very special meaning. It is well known (see, for instance, ref. 22) that each trajectory of the elliptical billiard has, on the configuration space, a conical caustic, confocal with the ellipse (Fig. 1).

If a segment of a given trajectory cuts the segment joining the two foci, all the other segments of this trajectory will cut it and the caustic will be a hyperbola. If it passes by one focus, the trajectory will always pass by the foci, thus having the two foci as a degenerate caustic. Otherwise, the caustic will be an ellipse. The number $|F(\varphi_0, \alpha_0)|^{1/2}$ measures the length of the minor axis of the conical caustic.

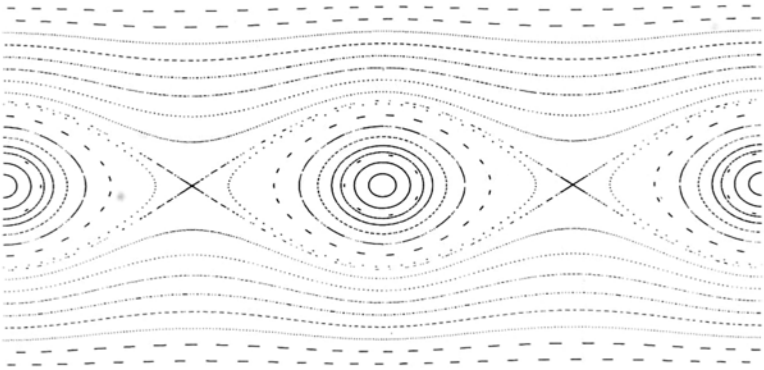

Fig. 2. Phase space of the elliptical billiard.

So, for every $a > 1$, the phase space $\mathscr{A}$ is foliated by the level curves of F (Fig. 2).

The orbits over $F = k > 0$ correspond to the trajectories with elliptical caustic. The trajectories over the major axis correspond to two hyperbolic 2-periodic orbits, with $F = 0$. They are connected by an invariant curve, corresponding to the trajectories that pass by the foci. It is a saddle connection and its level curve is $F = 0$. The trajectories over the minor axis are, on the phase space, the two elliptic 2-periodic orbits, with $F = 1 - a^2$ (except for $a = \sqrt{2}$, when they are parabolic). They are encircled by the curves $F = k$, $1 - a^2 < k < 0$, corresponding to the trajectories with hyperbolic caustic.

One of Poncelet's theorems on projective geometry (see, for instance, ref. 1) states that if, given two conics, there is a polygon having its vertices on one conic and sides tangent to the other, than there are infinitely many polygons with this same property. They all have the same number of sides and their vertices are ordered in the same way. This theorem can be translated, in the case of the elliptical billiard, stating that there are infinitely many trajectories sharing the same caustic and with the same dynamical behavior. On the phase space $\mathscr{A}$, this means that if a given orbit is periodic, all the other orbits on the same integral curve are also periodic, with the same period. If it is not periodic, the same will be true for all the orbits on the same integral curve. Also, on each integral curve, the points of the orbits are ordered in the same way. (For more information about Poncelet's theorem and billiards, see, for instance, refs. 5 and 23.)

To each integral curve $F = k$, or equivalently, to each conical caustic, is associated a rotation number.

- To $k > 0$ is associated a rotation number $\rho(k)$ such that:
  1. $0 < \rho < 1$.
  2. $\rho = n/p \in \mathbb{Q}$, $(n, p) \equiv 1$, corresponds to periodic orbits of period p, and $T^p(\varphi_0, \alpha_0) = (\varphi_0 + 2n\pi, \alpha_0) \equiv (\varphi_0, \alpha_0)$.
  3. $\rho \in \mathbb{R} \backslash \mathbb{Q}$ corresponds to dense orbits (dense on the level curve).
- To $1 - a^2 < k < 0$ is associated a rotation number $\tau(k)$ such that:
  1. $\nu(a) < \tau < 1$, where

$$\nu(a) = \begin{cases} 1 + \dfrac{1}{\pi} \arctan \dfrac{2\sqrt{a^2 - 1}}{a^2 - 2} & \text{if} \quad 1 < a < \sqrt{2} \\ 1/2 & \text{if} \quad a = \sqrt{2} \\ \dfrac{1}{\pi} \arctan \dfrac{2\sqrt{a^2 - 1}}{a^2 - 2} & \text{if} \quad \sqrt{2} < a \end{cases}$$

2. $\tau = n/p \in \mathbb{Q}$, $(n, p) \equiv 1$, corresponds to periodic orbits of period $2p$, crossing $2n$ times the minor axis of the ellipse.
3. $\tau \in \mathbb{R} \backslash \mathbb{Q}$ corresponds to dense orbits (dense on the level curve).

To define the function $\nu(a)$ we use that $(0, \pi/2)$ and $(\pi, \pi/2)$ are elliptic (or parabolic) fixed points of T^2, with eigenvalues

$$\lambda_j = e^{2\pi i \nu_j(a)}, \qquad \text{where} \quad \nu_j(a) = \frac{1}{\pi} \arctan \frac{\pm 2\sqrt{a^2-1}}{2-a^2}, \qquad j = 1, 2$$

Notice that $\nu(a)$ is strictly decreasing, $\nu(a) \to 1$ as $a \to 1^+$, and $\nu(a) \to 0^+$ as $a \to +\infty$. This indicates another difference between trajectories with elliptical caustic and those with hyperbolic caustic: for each $a > 1$, we have periodic orbits of every period with elliptical caustic, but not with hyperbolic caustic. First of all, the last ones have always even period. But, since $\tau = n/p > \nu(a)$, not all rotation numbers τ, and then not all periods, exist for every a. For instance, orbits of period 4 exist only for $a > \sqrt{2}$, and there is no period-8 orbit if $a < (4 - 2\sqrt{2})^{1/2}$. Figure 3 shows the bifurcation diagram for periodic orbits with hyperbolic caustic.

The fundamental structure of the phase space as displayed in Fig. 2 is invariant under scalings of the ellipse; in other words, it is the same for ellipses having the same eccentricity. However, as one can learn from the bifurcation diagram of Fig. 3, for different values of the eccentricity, the fine structure of the phase space around the elliptic 2-periodic points is quite different.

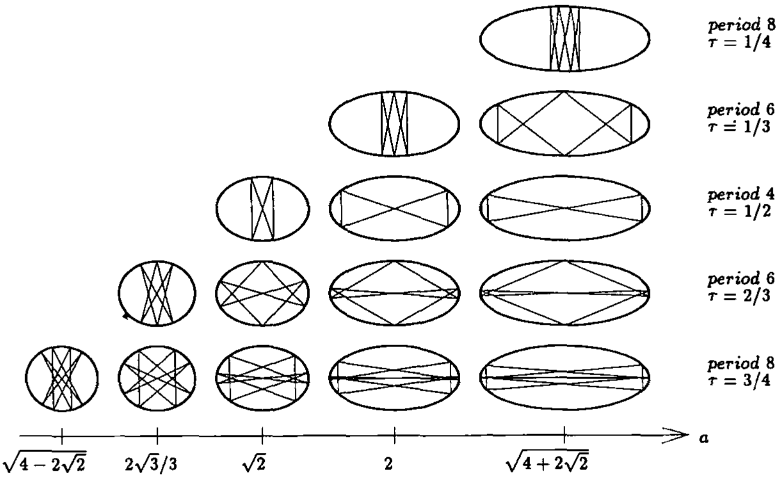

Fig. 3. Diagram of bifurcation of periodic orbits with hyperbolic caustic.

2. STATIC PERTURBATIONS OF THE ELLIPTICAL BILLIARD

Integrability, however, is not a robust property. KAM theory says that generic perturbations of the elliptical boundary will imply nonintegrability and the appearance of (perhaps very thin) chaotic zones on the phase space of the perturbed billiard. Many people have studied particular examples of these phenomena, getting explicit conditions for the transversality when the saddle connection is broken. Levallois and Tabanov,[14,24] for instance, have showed the nonintegrability for some one-parameter global symmetric perturbations (analytical and algebraic) of the ellipse, like the family

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1+2\varepsilon\left(\frac{x^2}{a^2}-\frac{x^4}{a^4}\right)+\varepsilon^2\left(\frac{x^4}{a^4}-\frac{x^6}{a^6}\right)$$

Levallois,[13] following Donnay, also studied local perturbations of the ellipse in the neighborhood of a point, changing the curvature but not the tangent at this point. And this problem is still nonintegrable.

Nevertheless, the structure of the phase space of the elliptical billiard is very strong. It is not so easy to go from absolute order (integrability) to total disorder (ergodicity). This can be seen in another example: the elliptical stadium.

The boundary of the elliptical stadium is constructed by joining two half-ellipses, with half-axes $a>1$ and $b=1$, by two segments of equal length $2h$ (Fig. 4).

The elliptical stadium billiard map is a homeomorphism, piecewise C^∞. It has two hyperbolic 2-periodic orbits (corresponding to the trajectories on the direction of the major axes of the half-ellipses), but the elliptic 2-periodic orbits of the elliptical billiard become an entire segment of 2-periodic points.

Canale and Markarian[4] proved the existence of symmetric periodic orbits for the elliptical stadium, of every even period, for every $1<a<\sqrt{2}$ and $h>0$ [they have the same shape as the periodic orbits with hyperbolic

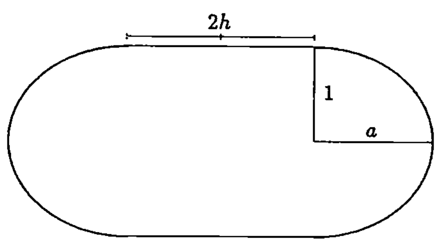

Fig. 4. Elliptical stadium.

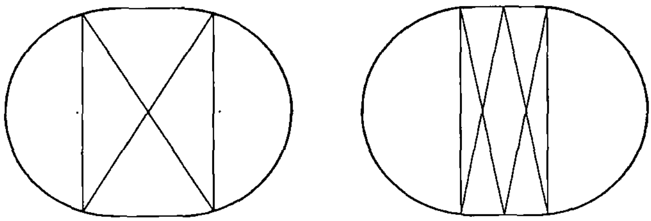

Fig. 5. Symmetric periodic orbits on the elliptical stadium.

caustic of the elliptical billiard, with period $2n$ and rotation number $\tau = (n-1)/n$] (see Figs. 3 and 5).

They pointed out that, for $1 < a < \sqrt{2}$, the 4-periodic orbit is elliptic if $h < (a^2-1)^{1/2}$ and is hyperbolic if $h > (a^2-1)^{1/2}$. The numerical simulations they carried out show that, while this 4-periodic orbit is elliptic, it is encircled by elliptic islands of positive measure (Fig. 6) and, fixing $a < \sqrt{2}$ and increasing h, those elliptic islands seem to be the last ones to disappear, among all those that can be observed, at least for small values of h.

Also, some numerical simulations we have carried out show that $\forall a > 1$ and $\forall h > 0$, initial conditions near the boundary of the phase space ($\alpha = 0$ or $\alpha = \pi$), after a finite number of iterations, come near the other boundary ($\alpha = \pi$ or $\alpha = 0$), showing the existence of what Mather[20] called glancing orbits, i.e., trajectories that change orientation in relation to the orientation of the boundary of the billiard. This indicates the existence of an invariant region that mixes up two invariant regions of the elliptical billiard. This phenomenon could be caused by the rupture of the saddle connection. Actually, the numerical simulations show that the invariant manifolds of the 2-periodic hyperbolic orbits of the elliptical stadium cross transversally and the problem is nonintegrable (Fig. 7).

Donnay[6] showed that if $1 < a < \sqrt{2}$ and if h is sufficiently large, then the elliptical stadium map has nonvanishing Lyapunov exponents almost everywhere, meaning sensitive dependence on initial conditions.

Markarian *et al.*[19] proved that if $a < (4-2\sqrt{2})^{1/2}$, then $\forall h > 2a^2(a^2-1)^{1/2}$, the elliptical stadium map is ergodic and has the K-property.

The proof of this result is based on the construction of a measurable eventually strictly invariant cone field on the phase space of the billiard map. Applying Wojtkowski's theorem,[25] it follows that for those values of

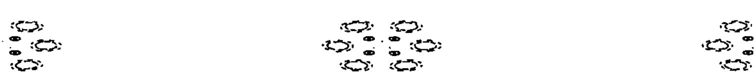

Fig. 6. Elliptical islands for $a = 1.07$ and $h = 0.05$.

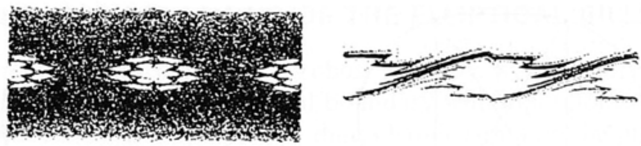

Fig. 7. 32,000 points of a unique orbit and the unstable manifold for $a = 1.07$ and $h = 0.05$.

a and h, the Lyapunov exponents are nonvanishing almost everywhere. We use the results of Liverani and Wojtkowski (ref. 17, §14.A) which assures ergodicity and, for instance, the results of Markarian (see ref. 18, Section 4) to prove that it is a K-system.

So we have to go "far" from the ellipse to have ergodicity if $a < \sqrt{2}$. And if $a > \sqrt{2}$, Bunimovich[3] conjectured that the elliptical stadium is not mixing. The numerical simulations suggest that, even for very big values of h, there are elliptic islands of positive measure that remain.

The persistence of the structure around the elliptic 2-periodic orbits (along the minor axis) can also be seen in a different stadium, proposed by Wojtkowski.[25] The ellipse is cut along the major axis and the two pieces are joined by straight segments of length $2h$. Two interior segments join the foci of the resulting half-ellipses (Fig. 8).

If $h < a^2 - 1$, there are elliptic islands of positive measure surrounding the elliptic 2-periodic points. If $h > a^2 - 1$, these islands disappear, giving rise to a Pesin component with nonvanishing Lyapunov exponents. In any case, independently of the value of h, two invariant regions with positive Lyapunov exponent exist, corresponding to orbits with elliptical caustic in the nonperturbed case. Then, if $h > a^2 - 1$, the phase space is neither integrable nor ergodic, but is decomposed into at least three ergodic components of positive measure.[18]

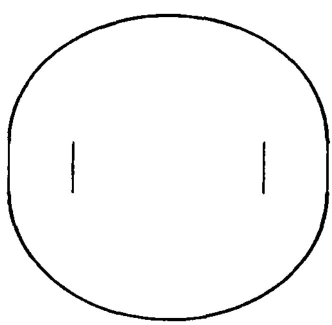

Fig. 8. Wojtkowski's stadium.

3. TIME-DEPENDENT PERTURBATIONS OF THE ELLIPTICAL BILLIARD

In this section, instead of studying a perturbation of the elliptical boundary, we are going to suppose that the ellipse deforms periodically with the time, remaining always an ellipse. A more detailed description of time-dependent billiards is given in refs. 8 and 9.

Billiards whose boundaries may deform are used as models in different branches of physics. The fundamental questions on this problem are whether a particle can reach unbounded velocities or whether a periodic motion of the boundary can give rise to random motions of the particle. The one-dimensional moving billiard problem, known as the Fermi accelerator, has been widely studied. A description of this and related one-dimensional models, as well as numerical results, may be found in the book by Lichtenberg and Liebermann[16] and in the references therein. Analytical results, also in dimension one, were proved by Pustyl'nikov,[21] Douady,[7] and Krüger *et al.*[10] Soft time-dependent billiards were studied by Levi[15] and Laederich and Levi.[12] Kutz and Zorn[11] studied numerically a rotating square billiard, through the adiabatic invariant.

We will suppose that, for every instant t, the elliptical boundary is given by $f(x, y, t) = x^2/a(t)^2 + y^2/b(t)^2 = 1$, where a and b are periodic functions of t, with commensurable periods.

Clearly, the billiard problem still has elastic collisions, but no reflections, since we have to take into account the normal velocity of the boundary at the impacts.

To build a mathematical model for this problem, let us take coordinates (x, y, z) on $\mathbb{R}^3$ and define the one-parameter Lagrangian system $L_\gamma = T_\gamma - V$, where

$$T_\gamma(x', y', z') = \frac{1}{2}\left(x'^2 + y'^2 + \frac{1}{\gamma^2} z'^2\right)$$

$$V(x, y, z) = \lim_{N \to \infty} [f(x, y, z)]^N$$

$$= \begin{cases} 0 & \text{inside the tube } (f < 1) \\ \infty & \text{outside the tube } (f > 1) \end{cases}$$

We thus have a rigid billiard on $\mathbb{R}^3$, inside the tube $f < 1$, whose trajectories are the geodesics of the metric given by T_γ. When γ goes to zero, those geodesics become straight lines and the solutions of this augmented billiard that have $z' = 1$ will project over those of the plane moving boundary billiard.

To see that they are well defined as a class of solutions, remark first that the equations of motion are

$$(x'', y'') = -\mathrm{grad}_{(x,y)} V(x, y, z)$$

$$z'' = -\gamma^2 \, \partial V/\partial z$$

Since the functions a and b are periodic, $\partial f/\partial z$ is bounded and so is $\partial V/\partial z$. Then, when $\gamma \to 0$, $z'' \to 0$. Choosing initial conditions $z' = 1$ and $z = 0$, $z(t) \to t$ when $\gamma \to 0$, or z is a "timelike" variable.

Suppose that a particle leaves the boundary $f = 1$ with velocity $\mathbf{w}_0(\gamma) = (a_0, b_0, c_0)$ and hits the boundary again. The outgoing velocity after the impact is $\mathbf{w}_1(\gamma) = \mathbf{w}_0(\gamma) - 2\langle \mathbf{w}_0(\gamma), \mathbf{n}_1 \rangle_\gamma \mathbf{n}_1$, where $\langle \, , \, \rangle_\gamma$ is the inner product inherited from T_γ and $\mathbf{n}_1$ is the unitary external normal vector with this inner product.

When $c_0 = 1$, $\lim_{\gamma \to 0} \mathbf{w}_1(\gamma) = (a_1, b_1, 1)$, or $\{z' = 1\}$ is an invariant section transverse to the flow defined by the Lagrangian, when $\gamma \to 0$.

Consider now the time-dependent billiard whose boundary at time t is given by the ellipse $C(t) = \{f(x, y, t) = 1\}$. Suppose that the particle leaves the boundary $C(t_0)$ from the point q_0 with velocity $\mathbf{v}_0 = (a_0, b_0)$. It reaches the boundary again at some future time t_1 at a point $q_1 \in C(t_1)$ with outgoing velocity $\mathbf{v}_1 = (a_1, b_1)$. If we look at the moving boundary as a surface in the (x, y, t) space, the trajectory of the particle at each instant will have velocity $\mathbf{w} = (\dot{x}, \dot{y}, 1)$. So the billiard map changes an initial condition $\mathbf{q}_0 = (q_0, t_0)$ with $\mathbf{w}_0 = (a_0, b_0, 1)$ into $\mathbf{q}_1 = (q_1, t_1)$ with $\mathbf{w}_1 = (a_1, b_1, 1)$.

It is easy to see that

$$u_i = \left. \frac{-\partial f/\partial t}{[(\partial f/\partial x)^2 + (\partial f/\partial y)^2]^{1/2}} \right|_{(q_i, t_i)}$$

is the normal velocity of the curve $C(t_i)$ at q_i. Simple calculations show that, as $\gamma \to 0$, $\mathbf{w}_1 = (a_1, b_1, 1) = (a_0, b_0, 1) - 2u_1 \boldsymbol{\eta}_1$, where $\boldsymbol{\eta}_1$ is the exterior normal vector of $C(t_i)$ at q_i. So, to calculate the next impact, all we have to know are t_0, q_0, and $\mathbf{v}_0$.

Since for each t, $C(t) = \{f(x, y, t) = 1\}$ is an ellipse, we parametrize it by the angle φ, as before. Working now on the plane, suppose that the particle leaves $q = (x(\varphi), y(\varphi)) \in C(t)$ with velocity $\mathbf{v} = (a, b)$. At q we have the orthonormal frame $\boldsymbol{\tau} = (\cos\varphi, \sin\varphi)$, the unitary tangent vector to $C(t)$, and $\boldsymbol{\eta} = (\sin\varphi, -\cos\varphi)$, the unitary exterior normal vector. In this frame $\mathbf{v} = (a, b) = v \cos\alpha \, \boldsymbol{\tau} - v \sin\alpha \, \boldsymbol{\eta}$, $v = \|\mathbf{v}\| = (a^2 + b^2)^{1/2}$. So, to give an initial condition we must give φ_0, α_0, v_0, and t_0. The next impact $(\varphi_1, \alpha_1, v_1, t_1)$

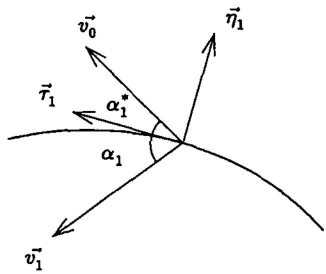

Fig. 9. Coordinates for the time-dependent billiard.

will be given by the formulas (obtained via elementary geometry and conservation of energy and momentum)

$$f(q_0+(t_1-t_0)\,\mathbf{v}_0,\,t_1)=1$$

$$\varphi_0+\alpha_0-\varphi_1+\alpha_1^*\equiv 0 \pmod{2\pi}$$

$$v_1\cos\alpha_1=v_0\cos\alpha_1^*$$

$$v_1\sin\alpha_1=v_0\sin\alpha_1^*-2u_1$$

The angle α_1^* is measured in the following way: the particle hits the boundary again at $t=t_1$ with velocity $\mathbf{v}_0$. In the new frame $(\boldsymbol{\tau}_1, \boldsymbol{\eta}_1)$ this vector is written $\mathbf{v}_0=v_0\cos\alpha_1^*\boldsymbol{\tau}_1+v_0\sin\alpha_1^*\boldsymbol{\eta}_1$ (Fig. 9).

Unfortunately, not all $(\varphi_0, \alpha_0, v_0, t_0)$ are admissible initial conditions. The movement must occur inside the region bounded by the deforming ellipse, or, analogously, inside the tube $f\leqslant 1$, on $\mathbb{R}^3$. Calling $\hat{t}$ the unitary vector in the t direction, we must ask that the outgoing velocity vector $\mathbf{v}_0+\hat{t}$ point inward, or equivalently, $\langle \mathbf{v}_0+\hat{t}, \nabla f_{(q_0,t_0)}\rangle$ is negative, or $v_0\sin\alpha_0+u_0>0$.

Let us call $\mathscr{B}=\{(\varphi_0, \alpha_0, v_0, t_0) \text{ s.t. } v_0\sin\alpha_0+u_0>0\}$ the set of the admissible initial conditions and T the map that to each $(\varphi_0, \alpha_0, v_0, t_0)\in\mathscr{B}$ associates the next impact $(\varphi_1, \alpha_1, v_1, t_1)$.

If f is C^k, $k\geqslant 2$, and avoiding a measure zero set of singularities, then $T\colon \mathscr{B}\to\mathscr{B}$ is a C^{k-2}-diffeomorphism that preserves the volume $dv=Rv(v\sin\alpha+u)\,d\varphi\,d\alpha\,dv\,dt$, where R is the radius of curvature of the ellipse $C(t)$ at φ.

The 4×4 Jacobian DT is composed of four 2×2 blocks A, B, C, and D, disposed as $\left(\begin{smallmatrix} A & D \\ C & B\end{smallmatrix}\right)$.

The "geometrical block"

$$A=\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22}\end{pmatrix}$$

corresponding to the (φ, α) coordinates may be written as

$$C_{11}=\frac{l_{01}}{R_1 \sin \alpha_1^*}-\frac{R_0 \sin \alpha_0}{R_1 \sin \alpha_1^*}+u_1 \mathcal{O}(1)$$

$$C_{12}=\frac{l_{01}}{R_1 \sin \alpha_1^*}+u_1 \mathcal{O}(1)$$

$$C_{21}=\frac{\sin \alpha_1^*}{\sin \alpha_1}\left(\frac{l_{01}-R_0 \sin \alpha_0-R_1 \sin \alpha_1^*}{R_1 \sin \alpha_1^*}\right)+u_1 \mathcal{O}(1)+\frac{\partial u_1}{\partial t_1} \mathcal{O}(1)+\frac{\partial u_1}{\partial \varphi_1} \mathcal{O}(1)$$

$$C_{22}=\frac{\sin \alpha_1^*}{\sin \alpha_1}\left(\frac{l_{01}-R_1 \sin \alpha_1^*}{R_1 \sin \alpha_1^*}\right)+u_1 \mathcal{O}(1)+\frac{\partial u_1}{\partial t_1} \mathcal{O}(1)+\frac{\partial u_1}{\partial \varphi_1} \mathcal{O}(1)$$

where $l_{01}=v_0(t_0-t_1)$ and R_i is the radius of curvature of $C(t_i)$ at φ_i, $i=1, 2$. As $\alpha_1^*=\alpha_1$, $u_1=\partial u_1/\partial t_1=\partial u_1/\partial \varphi_1=0$ in the rigid case, the (φ, α) block can be looked at as a perturbation of the rigid case.

What is surprising on this moving billiard map is that, in contrast to the rigid case, the angle $\alpha \in [-\pi, \pi)$. This can be seen, physically, for instance, if we suppose that $u_1>0$ (the boundary is moving outward) and the particle arrives with a normal incoming velocity $v_0 \sin \alpha_1^*$ such that $2u_1>v_0 \sin \alpha_1^*>u_1>0$. Then, $0>v_1 \sin \alpha_1>-u_1$ and $\alpha_1 \in [-\pi, 0)$. The particle hits the boundary and continues inside the billiard region, but outside $C(t_1)$. It does not rebound, but the boundary moves faster, in the normal direction.

The set of admissible initial conditions $\mathscr{B}$ is thus contained in the cylinder $[0, 2\pi) \times [-\pi, \pi) \times (0, +\infty) \times [0, P)$, where P is the common period of $a(t)$ and $b(t)$.

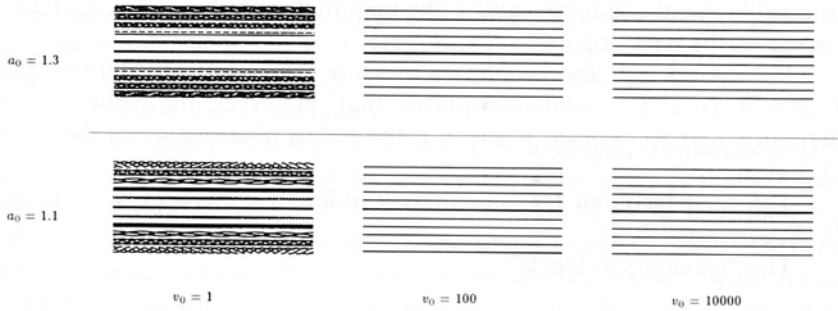

Fig. 10. Geometrical phase space for the breathing circle for $a_0=1.3$ and $a_0=1.1$.

Using this billiard diffeomorphism, we studied numerically some time-dependent ellipses, given by

$$f(x, y, t) = \frac{x^2}{a(t)^2} + \frac{y^2}{b(t)^2} = 1 \qquad \text{with} \quad a(t) = a_0 + \varepsilon \cos \omega t$$

In this paper we will present (i) *the breathing circle*, where $b(t) = a(t)$, (ii) *the confocal motion*, where $a^2(t) - b^2(t) = a_0^2 - b_0^2$, (iii) *the dipole*, where $b(t) = b_0$, and (iv) *the quadrupole*, where $b(t) = b_0 - \varepsilon \cos(\omega t)$, for a few values of a_0 and b_0, fixing $\varepsilon = 0.1$ and $\omega = 1$.

In each example, we investigate the consequences of the time-dependent perturbation on the phase space of the elliptical billiard [corresponding to the (φ, α) coordinates on the four-dimensional phase space of the moving ellipse] and the time evolution of the velocity.

Note that in all the examples above and for each t the boundary is either an ellipse centered at the origin, with half-axis changing in size, but staying always along the x and y directions, or a circle centered at the origin. To each one of these curves is associated a billiard with its two-dimensional phase space. The fundamental structure of these phase spaces for the ellipses is described in Section 1 and displayed in Fig. 2. The phase space of the circular billiard is just foliated by horizontal invariant lines.

Figures 10–13 present "geometrical" phase spaces (the $\varphi \times \alpha$ diagrams) for different values of the initial velocity v_0. For each v_0 we take a few initial values of φ_0 and α_0, fixing $t_0 = 0$.

Figure 10 shows the geometrical phase space for the breathing circle, with $a_0 = 1.3$ and $a_0 = 1.1$. It is clear that the structure of the static circular billiard phase space is maintained and that the results do not depend on a_0.

Figure 11 shows the geometrical phase space for the confocal motion with $a_0 = 1.3$, $b_0 = 1.0$ and $a_0 = 1.1$, $b_0 = 1.0$. For every a_0, the fundamental

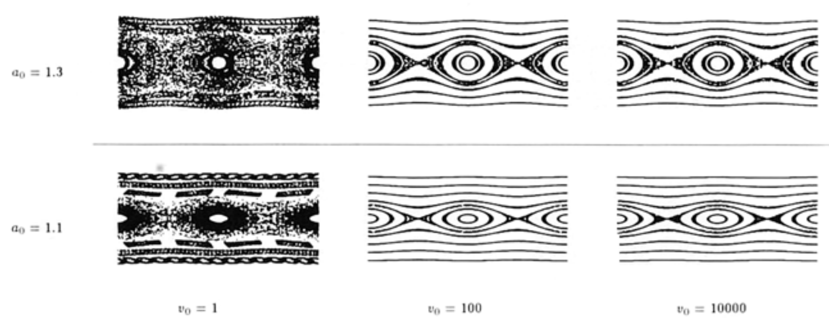

Fig. 11. Geometrical phase space for the confocal motion for $a_0 = 1.3$, $b_0 = 1.0$ and $a_0 = 1.1$, $b_0 = 1.0$.

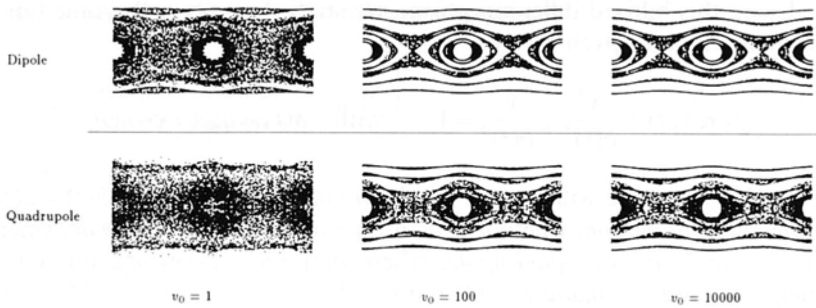

Fig. 12. Geometrical phase space for the dipole and the quadrupole for $a_0 = 1.3$, $b_0 = 1.0$.

structure of phase space of the static elliptical billiard is present, but the map is clearly nonintegrable.

In the breathing circle as in the confocal motion, the greater the velocity, the more similar to the static case is the $\varphi \times \alpha$ diagram. For small values of the velocity, it appears more disordered, looking like a perturbation of the geometrical phase space for large values of the velocity.

Figure 12 displays the geometrical phase space for the dipole and the quadrupole, with $a_0 = 1.3$, $b_0 = 1.0$. Here we still find the fundamental structure of the static elliptical billiard and the nonintegrability. Note that for this choice of a_0 and b_0 the major axis of each ellipse is always on the x axis and the minor one on the y axis, and so the global structure of the phase space of each instantaneous ellipse is the same; in particular, the stability of the 2-periodic orbits on the diameters is unchanged.

A quite different situation appears when $a_0 = 1.1$ and $b_0 = 1.0$, for the dipole and the quadrupole. Here the instantaneous ellipse degenerates into

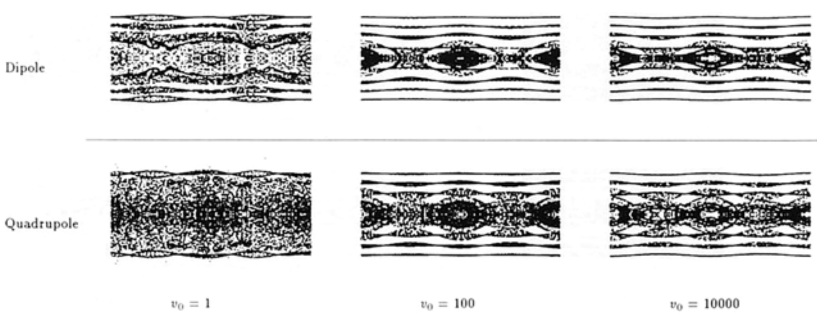

Fig. 13. Geometrical phase space for the degenerate dipole and the degenerate quadrupole for $a_0 = 1.1$, $b_0 = 1.0$.

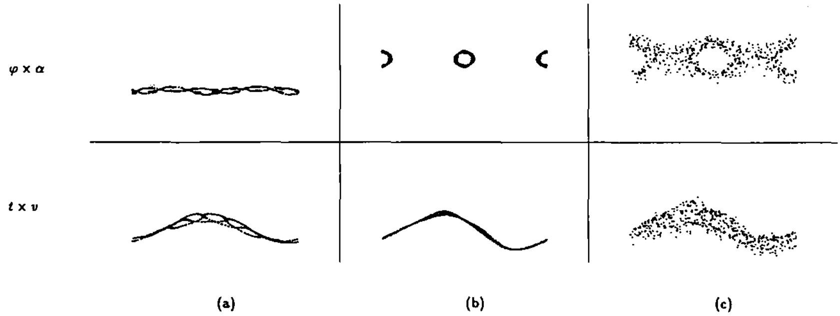

Fig. 14. Behavior of an orbit with initial condition near (a) the boundary, (b) the elliptic periodic orbit, and (c) the broken saddle connection, for $v_0 = 1$.

a circle. The corresponding geometrical phase spaces look more like a perturbation of the circular billiard than of the elliptical one, as shown in Fig. 13.

For these degenerate dipole and quadrupole, as the half-axes are interchanged, the geometrical phase space is like an overlap of elliptical billiard phase spaces (at least for large velocities). A reasonable guess would be that this overlap also occurs in the nondegenerate cases of Fig. 12.

Since the x axis and the y axis are diameters of the ellipses for every t, Douady's theorem (ref. 7, Chapter 3,IV) may be applied, implying the boundedness of the velocities for the dynamics restricted to these diameters. Figures 14 and 15 show what happens for other initial conditions for the confocal motion with $a_0 = 1.4$ and $b_0 = 1.0$. Remark that the more irregular behavior occurs for initial conditions near the broken saddle connection, with a corresponding spreading of the velocities (Fig. 14). Figure 15 displays the $t \times v$ diagram for initial conditions near the broken saddle connection for the same model and different initial velocities.

Trying the other examples, we observed that the behavior of their $t \times v$ diagrams is very similar to that of the corresponding diagrams for the confocal motion. So, we do not display them here.

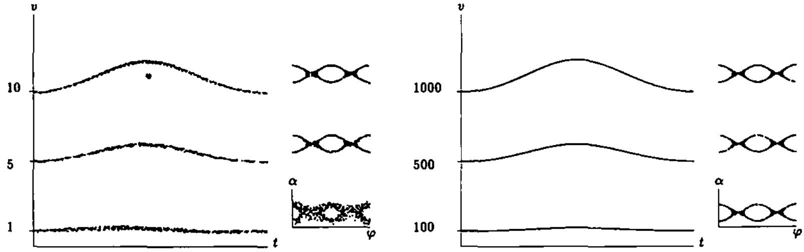

Fig. 15. Time evolution of the velocity.

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REFERENCES

1. W. P. Barth, Elliptic moduli curves and Poncelet polygons, Advanced Workshop on Algebraic Geometry, ICTP, Trieste, Italy (1994).
2. M. V. Berry, Regularity and chaos in classical mechanics, illustrated by three deformations of a circular billiard, *Eur. J. Phys.* **2**:91–102 (1981).
3. L. A. Bunimovich, On absolutely focusing mirrors, in *Ergodic Theory and Related Topics*, U. Kringel *et al.*, eds. (Springer, Berlin, 1993); L. A. Bunimovich, Conditions on stochasticity of two-dimensional billiards, *Chaos* **1**:187–193 (1991).
4. E. Canale and R. Markarian, Simulación de billares planos, in *Anales IEEE, Segundo Seminario de Informática en el Uruguay* (1991), pp. 71–96; and private communication.
5. S.-J. Chang and R. Friedberg, Elliptical billiards and Poncelet's theorem, *J. Math. Phys.* **29**(7):1537–1550 (1988).
6. V. J. Donnay, Using integrability to produce chaos: Billiards with positive entropy, *Commun. Math. Phys.* **141**:225–257 (1991).
7. R. Douady, Applications du théorème des tores invariants, Thèse de 3ème Cycle, Université Paris VII (1982).
8. J. Koiller, R. Markarian, S. Oliffson Kamphorst, and S. Pinto de Carvalho, A geometric framework for time-dependent billiards, in *New Trends in Hamiltonian Systems* (World Scientific, Singapore, to appear).
9. J. Koiller, R. Markarian, S. Oliffson Kamphorst, and S. Pinto de Carvalho, Time-dependent Billiards, Preprint (1994); *Nonlinearity*, to appear.
10. T. Krüger, L. D. Pustyl'nikov, and S. E. Troubetzkoy, Acceleration of bouncing balls in external fields, *Nonlinearity* **8**:397–410 (1995).
11. N. Kutz and E. Zorn, Adiabatische Invarianten und Hannay-Winkel im reichteckigen Billard, Studienarbeit, Technischen Universität Berlin (1988).
12. S. Laederich and M. Levi, Invariant curves and time dependent potentials, *Ergodic Theory Dynam. Syst.* **11**:365–378 (1991).
13. P. Levallois, Non-integrabilité des billiards définis par certaines perturbations algébriques d'une ellipse et du flot géodesique de certaines perturbations algébriques d'un ellipsoide, Thèse, Université Paris VII (1993).
14. P. Levallois and M. B. Tabanov, Séparation des séparatrices du billiard elliptique pour une perturbation algébrique et symétrique de l'ellipse, *C. R. Acad. Sci. Paris I* **316**:589–592 (1993).
15. M. Levi, Quasiperiodic motions in superquadratic time-periodic potentials, *Commun. Math. Phys.* **143**:43–83 (1991).
16. A. J. Lichtenberg and M. A. Liebermann, *Regular and Stochastic Motion* (Springer-Verlag, Berlin, 1983).
17. C. Liverani and M. Wojtkowski, Ergodicity in Hamiltonian systems, SUNY, Stony Brook, Preprint (1992).

18. R. Markarian, New ergodic billiards: Exact results, *Nonlinearity* **6**:819–841 (1993).
19. R. Markarian, S. Oliffson Kamphorst, and S. Pinto de Carvalho, Chaotic properties of the elliptical stadium, preprint (1994); *Commun. Math. Phys.*, to appear.
20. J. Mather, Glancing billiards, *Ergodic Theory Dynam. Syst.* **2**:397–403 (1982).
21. L. D. Pustyl'nikov, Stable and oscillating motions in nonautonomous dynamical systems II. *Trans. Moscow Math. Soc.* **2**:1–101 (1978).
22. Ya. G. Sinai, *Introduction to Ergodic Theory* (Princeton University Press, Princeton, New Jersey, 1976).
23. S. Tabachnikov, Billiards, Preprint, University of Arkansas (1994); in "*Panoramas et Synthèses*," SMF, to appear.
24. M. B. Tabanov, Splitting of separatrices for Birkhoff's billiard under symmetrical perturbation of an ellipse, in *Rencontres Franco-Russes de Géometrie* (CIRM, France, 1992).
25. M. Wojtkowski, Principles for the design of billiards with non vanishing Lyapunov exponents, *Commun. Math. Phys.* **105**:391–414 (1986).